SINGULAR MEASURES AND HAUSDORFF MEASURES

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ABSTRACT
An example is given of a family of singular probability measures on the unit interval which are supported on a set of fractional Hausdorff dimension but cannot be represented as Hausdorff measures.

Let $x = .x_1x_2\cdots$ be the m-adic expansion of a number in the unit interval. $x_i = x_i(x)$ are Borel measurable random variables. Any probability measure P on the Borel field produces a stochastic process $\{x_n\}$ and vice versa.

An interesting connection was established in [1], [2], [5] between the entropy of the stochastic process $\{x_n\}$ and the support of P. The following theorem can be easily derived from these papers using Breiman's ergodic theorem [3]. Denote by $\dim(A)$ the Hausdorff dimension of the set A in the unit interval.

THEOREM *If P is a probability measure such that $\{x_i\}$ is stationary ergodic process with relative entropy h (entropy in $\log m$ units), then*

a) *There exists a measurable set E such that $P(E) = 1$ and $\dim(E) = h$.*

b) *If $\dim(F) < h$ then $P(F) = 0$.*

In view of this Theorem we raise the following question: Can a set E be selected so that $P(A) =$ const. $\mu_h(A \cap E)$ where $\mu_h(B)$ is the Hausdorff h-measure of B? Or more generally,

$$(*) \qquad P(A) = \text{const. } \mu_\psi(A \cap E)$$

where $\psi = \psi(t)$ is a real function on $0 \leqq t \leqq t_0$, for some $t_0 > 0$, continuous, concave and strictly increasing such that $h(0) = 0$. $\mu_\psi(B)$ is defined as follows. Let $\{I_\alpha] = \mathscr{C}_\rho$ be a sequence of intervals of length less than ρ covering A.

$$\mu_\psi(A) = \lim_{\rho \to 0} \inf_{\mathscr{C}\rho} \sum_{I\alpha \in \mathscr{C}\rho} \psi(|I_\alpha|).$$

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(h-measure is the case where $\psi(t) = t^h$). In some cases, like the Cantor measure (the measure which makes x_i, in the ternary expression, i.i.d. $P(x_i = 0) = P(x_i = 2) = \frac{1}{2}$) the answer is yes. The purpose of this note is to show that this is an exceptional case by giving an example of a family of singular probability measures for which the answer is negative. Let P be the measure which makes x_i i.i.d. in the binary expansion $P(x_i = 1) = p$, $P(x_i = 0) = 1-p$, $0 < p < \frac{1}{2}$. In what comes let $\log x = \log_2 x$. Put

$$y_i = \begin{cases} -\log p & \text{if } x_i = 1 \\ \log(1-p) & \text{if } x_i = 0 \end{cases}$$

then y_i are i.i.d., $E(y_i) = h =$ the entropy of $\{x_i\}$ and $\mathrm{Var}(y_i) = \sigma^2 > 0$. Let I_n^α denote a binary interval of order n and $I_n(x)$ the binary interval of order n containing x. Obviously $S_n(x) = \Sigma_{i=1}^n(x) = -\log P(I_n(x))$. Let us prove first that there exists a Borel set A such that $P(A) = 1$ and $\mu_h(A) = 0$ (and therefore P cannot be represented in the form (*) for $\psi(t) = x^h$). The set

$$A = \{S_n \leqq nh - \sigma\sqrt{n}\ \text{i.o.}\} = \{x : P(I_n(x)) \geqq 2^{-nh+\sigma\sqrt{n}}\text{i.o.}\}$$

has probability one. For a given n_0 choose for each $x \in A$ the first $I_n(x)$ $n \geqq n_0$ such that $P(I_n(x)) \geqq 2^{-nh+\sigma\sqrt{n}}$. The sequence of intervals $\{I_n^\alpha\}$ thus obtained form a covering of A with $\rho = \alpha^{-n_0}$. Therefore

$$1 = \Sigma P(I_n^\alpha) \geqq \sum_\alpha |I_n^\alpha|^h 2^{\sigma\sqrt{n}} \geqq 2^{\sigma\sqrt{n^0}} \Sigma |I_n^\alpha|^h.$$

letting n_0 tend to infinity we get a vanishing sequence of coverings.

Now, suppose $\psi(t)$ is a continuous concave function near the origin such that $\psi(0) = 0$. Then we prove that either

a) There exist a set A, $P(A) = 1$, $\mu_\psi(A) = 0$, or

b) For every set A such that $P(A) > 0$, $\mu_\psi(A) = \infty$.

Let $\phi(n)$ be a sequence such that

$$\psi(2^{-n}) = 2^{-nh+\sigma\sqrt{n}\phi(n)}$$

in view of what we have proved and the theorem we have to consider only ψ such that $\phi(n)$ in increasing sequence and $\phi(n/\sqrt{n} \to 0$. For such sequences the law of iterated logarithm asserts, [4], that either $\phi(n)$ belongs to the upper class or to the lower class with respect to $\{y_i\}$. In either case $\phi(n)$ + const.$/\phi(n)$ belongs to the same class ([4] page 383, remark 1). Suppose $\phi(n)$ is in the lower class, then so is $\phi(n) + 1/\phi(n)$ and

$$P\left\{S_n \leqq nh - \sqrt{n}\,\sigma(\phi(n) + \frac{1}{\phi(n)}\text{i.o.}\right\}$$

$$= P\{x: P(I_n(x)) \geqq 2^{-nh+\sigma\sqrt{n}(\phi(n)+1/\phi(n))}\text{ i.o.}\} = 1.$$

Therefore we can select a covering $\{I_n^\alpha\}$ of this set such that $n \geqq n_0$ and

$$1 = \Sigma P(I_n^\alpha) \geqq \sum_\alpha \psi(|I_n^\alpha|)2,^{(\sqrt{n}/\phi(n))} \geqq 2,^{(\sqrt{n^0}/\phi(n^0)} \Sigma\psi(|I_n^\alpha|)$$

but $(\sqrt{n_0})/\phi(n_0) \to \infty$, therefore $\mu_\psi(A) = 0$. Now if $\phi(n)$ belongs to the upper class then so does $\phi(n) - 1/\phi(n)$ and

$$P\{x: P(I_n(x)) \geqq 2^{-nh+\sigma\sqrt{n}(\phi(n)-1/\phi(n))}\text{ i.o.}\} = 0.$$

Thus the sequence of sets

$$E_m = \{x: P(I_n(x)) \leqq 2^{-nh+\sigma\sqrt{n}(\phi(n)-1/\phi(n))},\ n \geqq m\}$$

is monotonically increasing and $\lim P(E_m) = 1$. Given A with positive probability and $0 < \delta < P(A)$ there is an m such that $P(A \cap E\ ) \geqq \delta$. For any covering $\{I_n^\alpha\}$ (According to Billingsley [1] it is enough to consider binary coverings) of $A \cap E_m$ such that $n \geqq m$ for all α

$$2^{-},^{(\sqrt{m}/\phi(m\ )}\cdot\sum_\alpha \psi(|I_n^\alpha|) \geqq \sum_\alpha \psi(|I_n^\alpha|)2^{-\sigma\sqrt{n}/\phi(n)} \geqq \sum_\alpha P(I_n^\alpha) \geqq P(A \cap E_m) \geqq \delta$$

and therefore as $m \to \infty$, $\Sigma\psi(|I_n^\alpha|) \to \infty$.

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